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Solution of a System of Linear Fredholm Integral Equations of the Second Kind by Iteration Methods

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Abstract

In this paper, we consider the linear system of Fredholm integral equation of the second kind. Three methods are used to solve this system, successive approximation method, Aitken's method depending on successive approximation method and a new procedure which is Aitken's method depending on Adomian decomposition method. A comparison between approximate and exact results for two numerical examples depending on the least-square error, are given to show the accuracy of the results obtained by using these methods.

Introduction

Recently, (Saeed, 2006) used iteration methods for solving linear system of Volterra integral and integro differential equations.(Babolian et al., 2004), used the Adomian decomposition method for solving linear system of Fredholm integral equations of the second kinds (LSFIEs,2nd). Also (Delves & Mohamed, 1985), uses trapezoid method for solving 2×2 linear systems of Fredholm integral equations of the second kind.

In this paper, we reformulate successive approximation method (Jerri, 1985) for solving LSFIEs,2nd . In section 3, we apply Aitken's method (Kincaid & Cheny, 2002) depending on successive approximation method to solve LSFIEs,2nd. We also derive a procedure in section 4 for solving LSFIEs,2nd depending on Aitken's method and Adomian decomposition method.

LSFIEs,2nd can be written in the following form (Babolian, et al. 2004):

$$u_i(x) = f_i(x) + \sum_{j=1}^{n}\int_a^b k_{ij}(x,t)u_j(t)dt, \quad x \in [a,b], \qquad \ldots(1)$$

where $f_i(x)$, $k_{ij}(x,t)$, i, j=1, 2, …, n are known continuous functions and $u_i(x)$, i=1, 2, …, n are unknown functions.

Also the Aitken's formula (Kincaid & Cheny, 2002) is given as follows:

$$\hat{v}_k = \frac{v_k v_{k+2} - v_{k+1}^2}{v_{k+2} - 2v_{k+1} + v_k}, \; k=0, 1, 2, \ldots$$

Successive approximation method for solving equation (1)

Successive approximation method or solving equation (1) can be summarized by the following: Let

$$u_{i0}(x) = u_0(x) = 0, \qquad \ldots(2.a_0)$$

$$u_{i1}(x) = f_i(x) + \sum_{j=1}^{n}\int_a^b k_{ij}(x,t)u_0(t)dt = f_i(x), \qquad \ldots(2.a_1)$$

$$u_{i2}(x) = f_i(x) + \sum_{j=1}^{n}\int_a^b k_{ij}(x,t)u_{j1}(t)dt, \qquad \ldots(2.a_2)$$

This process can be continued to obtain the m^{th} approximation,

$$u_{im}(x) = f_i(x) + \sum_{j=1}^{n}\int_a^b k_{ij}(x,t)u_{j,m-1}(t)dt,\ i=1, 2,\ldots,n;\ m=1, 2,\ldots \quad (2.a_{m+1})$$

We see that the sequence $u_{im}(x)$ will converge to the solution $u_i(x)$ of equation (1) as $m \to \infty$ for i=1, 2, …, n.

Aitken on successive approximation method for solving equation (1)

In this section, we apply Aitken's method depending on successive approximation method for solving equation (1) as follows:

In equations (2.a$_0$), (2.a$_1$) and (2.a$_2$) find $u_{i0}(x)$, $u_{i1}(x)$ and $u_{i2}(x)$ respectively and substitute these values in Aitken's formula we get

$$u_i(x) = \frac{u_{i0}(x)u_{i2}(x) - u_{i1}^2(x)}{u_{i2}(x) - 2u_{i1}(x) + u_{i0}(x)},\ i=1, 2, \ldots, n. \qquad \ldots(3)$$

Then equation (3) is a new procedure for finding the approximate (sometime exact, depending on the number of iterations) solution of equation (1) by using only three function values. We may use any three successive values $u_{il}(x)$, $u_{i,l+1}(x)$ and $u_{i,l+2}(x)$, l=0, 1, … in equation (3), we get

$$u_i(x) = \frac{u_{il}(x)u_{i,l+2}(x) - u_{i,l+1}^2(x)}{u_{i,l+2}(x) - 2u_{i,l+1}(x) + u_{il}(x)},\ i=1, 2, \ldots,n,\ l=0, 1, 2, \qquad \ldots(4)$$

In equation (4) we get the exact solution for sufficiently large l.

Aitken on Adomian decomposition method for solving equation (1)

In this section, for the first time Aitken's method has been used successfully on Adomian decomposition method as extrapolated formula to find approximate (sometime exact, depending on the number of iterations) solution of equation (1) where four successive approximated values of the unknown function $u_i(x)$ are found to activate this method by using Adomian decomposition method as follows:

Step 1: Recall equation (1): $u_i(x) = f_i(x) + \sum_{j=1}^{n}\int_a^b k_{ij}(x,t)u_j(t)dt,$

Step 2: We find $u_{i0}(x)$, $u_{i1}(x)$, $u_{i2}(x)$ and $u_{i3}(x)$ as follows:

$$u_{i0}(x)=f_i(x),$$

$$u_{i1}(x)=\sum_{j=1}^{n}\int_a^b k_{ij}(x,t)u_{j0}(t)dt,$$

$$u_{i2}(x)=\sum_{j=1}^{n}\int_a^b k_{ij}(x,t)u_{j1}(t)dt,$$

$$u_{i3}(x)=\sum_{j=1}^{n}\int_a^b k_{ij}(x,t)u_{j2}(t)dt.$$

Step 3: Let first approximation to the exact solution is:

$$\alpha_{i0}(x) = u_{i0}(x) + u_{i1}(x), \qquad \ldots(5)$$

the second approximation is

$$\alpha_{i1}(x) = u_{i0}(x) + u_{i1}(x) + u_{i2}(x), \qquad \ldots(6)$$

and the third approximation is

$$\alpha_{i2}(x) = u_{i0}(x) + u_{i1}(x) + u_{i2}(x) + u_{i3}(x). \qquad \ldots(7)$$

We can simplify equations (6) and (7) we obtain:

$$\alpha_{i1}(x) = \alpha_{i0}(x) + u_{i2}(x), \qquad \ldots(8)$$

$$\alpha_{i2}(x) = \alpha_{i1}(x) + u_{i3}(x). \qquad \ldots(9)$$

Step 4: Suppose that the exact solution of equation (1) is $\alpha_i(x)$.

Step 5: Use equations (5), (8) and (9) as an approximated solution and substituting them in Aitken's formula, we get

$$\alpha_i(x) \cong \frac{\alpha_{i0}(x)\alpha_{i2}(x) - \alpha_{i1}^2(x)}{\alpha_{i2}(x) - 2\alpha_{i1}(x) + \alpha_{i0}(x)}. \qquad \ldots(10)$$

Indeed equation (10), identifies a new procedure to find an approximate value of the exact solutions $u_i(x)$, i=1, 2, ..., n by using only three function values $\alpha_{i0}(x)$, $\alpha_{i1}(x)$ and $\alpha_{i2}(x)$, in general we get:

$$\alpha_i(x) = \frac{\alpha_{im}(x)\alpha_{i,m+2}(x) - \alpha_{i,m+1}^2(x)}{\alpha_{i,m+2}(x) - 2\alpha_{i,m+1}(x) + \alpha_{im}(x)}, \quad i=1, 2, 3, \ldots, n;\ m=0, 1, 2, \ldots.$$

Equality holds for sufficiently large m.

Numerical examples

In this section, two new examples are presented to illustrate these considerations:

Example 1: Consider the following system of linear integral equations of the second kind with the exact solutions $u_1(x) = e^x$ and $u_2(x) = 1 - x$:

$$u_1(x) = e^x - x - \frac{1}{6}x^2 + \int_0^1 x t u_1(t)dt + \int_0^1 x^2 u_2(t)dt$$

$$u_2(x) = \frac{5}{6} - x + \int_0^1 t u_2(t)dt \; .$$

Example 2: Consider the following system of linear integral equations of the second kind with the exact solutions $u_1(x) = 2x$ and $u_2(x) = 3x^2$:

$$u_1(x) = \frac{3}{4} + \int_0^1 x u_1(t)dt + \int_0^1 (x-t)u_2(t)dt$$

$$u_2(x) = -\frac{1}{12} - x + 3x^2 + \int_0^1 (x-t)u_1(t)dt + \int_0^1 t u_2(t)dt \; .$$

Note: L.S.E.=least square error.

SAM=Successive approximation method.

ADM=Adomian decomposition method.

Table (1) the results for the approximate solution $u_1(x)$ of Example 1, by using SAM

x	Exact value of $u_1(x)$	Approximate value of $u_1(x)$ by		
		3 iterations	6 iterations	12 iterations
0	1	1	1	1
0.1	1.105170918	1.099059807	1.104651371	1.105164267
0.2	1.221402758	1.208763869	1.22031158	1.221388641
0.3	1.349858808	1.330275474	1.348143916	1.349836412
0.4	1.491824698	1.464880253	1.489434008	1.491793209
0.5	1.648721271	1.613999048	1.645602701	1.648679875
0.6	1.822118800	1.779202134	1.818220266	1.822066684
0.7	2.013752707	1.962224930	2.009022126	2.013689057
0.8	2.225540928	2.164985373	2.219926217	2.225464930
0.9	2.459603111	2.389603111	2.453052185	2.459513950
1	2.718281828	2.638420717	2.710742605	2.718178692
L.S.E.		2.2×10^{-2}	1.887×10^{-4}	3.458×10^{-8}

Table (2) the results for the approximate solution u2(x) of Example 1, by using SAM

x	Exact value of $u_2(x)$	Approximate value of $u_2(x)$ by		
		3 iterations	6 iterations	12 iterations
0	1	1	0.997395833	0.99995931
0.1	0.9	0.979166667	0.897395833	0.89995931
0.2	0.8	0.879166667	0.797395833	0.79995931
0.3	0.7	0.779166667	0.697395833	0.69995931
0.4	0.6	0.679166667	0.597395833	0.59995931
0.5	0.5	0.579166667	0.497395833	0.49995931
0.6	0.4	0.479166667	0.397395833	0.39995931
0.7	0.3	0.379166667	0.297395833	0.29995931
0.8	0.2	0.279166667	0.197395833	0.19995931
0.9	0.1	0.179166667	0.097395833	0.09995931
1	0	0.079166667	-0.002604167	-0.00004069
L.S.E.		4.8×10^{-3}	7.460×10^{-5}	1.821×10^{-8}

Table (3) the results for the approximate solution $u_1(x)$ of Example 1, by using Aitken's on SAM and ADM

x	Exact value of $u_1(x)$	Approximate value of $u_1(x)$ by	
		Aitken's on SAM	Aitken's on ADM
0	1	1	1
0.1	1.105170918	1.103782029	1.104529892
0.2	1.221402758	1.218481796	1.220059946
0.3	1.349858808	1.345266971	1.347756004
0.4	1.491824698	1.485427391	1.488906111
0.5	1.648721271	1.640387937	1.644933392
0.6	1.822118800	1.811722761	1.817410280
0.7	2.013752707	2.001171008	2.008074241
0.8	2.225540928	2.210654197	2.218845150
0.9	2.459603111	2.442295419	2.451844490
1	2.718281828	2.698440559	2.709416580
L.S.E.		1.3×10^{-3}	2.675×10^{-4}

Table (4) the results for the approximate solution $u_2(x)$ of Example 1, by using Aitken's on SAM and ADM

x	Exact value of $u_2(x)$	Approximate value of $u_2(x)$ by	
		Aitken's on SAM	Aitken's on ADM
0	1	1	1
0.1	0.9	0.9	0.9
0.2	0.8	0.8	0.8
0.3	0.7	0.7	0.7
0.4	0.6	0.6	0.6
0.5	0.5	0.5	0.5
0.6	0.4	0.4	0.4
0.7	0.3	0.3	0.3
0.8	0.2	0.2	0.2
0.9	0.1	0.1	0.1
1	0	0	0
L.S.E.		0	0

Table (5) the result for the approximate solution of $u_1(x)$ of Example 1, by using ADM derived by (Babolian et al., 2004)

x	Exact value of $u_1(x)$	Approximate value of $u_1(x)$ after		
		3 iterations	6 iterations	12 iterations
0	1	1	1	1
0.1	1.105170918	1.102578325	1.104928291	1.105167616
0.2	1.221402758	1.216009240	1.220891463	1.221395747
0.3	1.349858808	1.341456030	1.349052802	1.349847680
0.4	1.491824698	1.480204327	1.490697940	1.491809047
0.5	1.648721271	1.633674974	1.647247720	1.648700690
0.6	1.822118800	1.803438245	1.820272414	1.822092883
0.7	2.013752707	1.991229559	2.011507444	2.013721047
0.8	2.225540928	2.198966854	2.222870747	2.225503117
0.9	2.459603111	2.428769778	2.456481969	2.459558742
1	2.718281828	2.682980903	2.714683685	2.718230495
L.S.E.		4.2×10^{-3}	4.268×10^{-5}	8.560×10^{-9}

Table (6) the result for the approximate solution of $u_2(x)$ of Example 1, by using ADM derived in (Babolian et al., 2004)

x	Exact value of $u_2(x)$	Approximate value of $u_2(x)$ after		
		3 iterations	6 iterations	12 iterations
0	1	0.989583333	0.998697917	0.999979655
0.1	0.9	0.889583333	0.898697917	0.899979655
0.2	0.8	0.789583333	0.798697917	0.799979655
0.3	0.7	0.689583333	0.698697917	0.699979655
0.4	0.6	0.589583333	0.598697917	0.599979655
0.5	0.5	0.489583333	0.498697917	0.499979655
0.6	0.4	0.389583333	0.398697917	0.399979655
0.7	0.3	0.289583333	0.298697917	0.299979655
0.8	0.2	0.189583333	0.198697917	0.199979655
0.9	0.1	0.089583333	0.098697917	0.099979655
1	0	-0.010416667	-0.001302083	-0.000020345
L.S.E.		1.2×10^{-3}	1.865×10^{-5}	4.553×10^{-9}

Table (7) shows the comparison between least square errors for SAM and ADM which is derived by (Babolian et al., 2004) for Example 1

x	3 iterations		6 iterations		12 iterations	
	$u_1(x)$	$u_2(x)$	$u_1(x)$	$u_2(x)$	$u_1(x)$	$u_2(x)$
SAM	2.2×10^{-2}	4.8×10^{-3}	1.887×10^{-4}	7.460×10^{-5}	3.458×10^{-8}	1.821×10^{-8}
ADM	4.2×10^{-3}	1.2×10^{-3}	4.268×10^{-5}	1.865×10^{-5}	8.560×10^{-9}	4.553×10^{-9}

Table (8) the results for the approximate solution $u_1(x)$ of Example 2, by using SAM

x	Exact value of $u_1(x)$	Approximate value of $u_1(x)$ after		
		3 iterations	6 iterations	12 iterations
0	0	0.031250000	-0.000434028	0.000000251
0.1	0.2	0.225694444	0.199681713	0.200000184
0.2	0.4	0.420138889	0.399797454	0.400000117
0.3	0.6	0.614583333	0.599913194	0.600000050
0.4	0.8	0.809027778	0.800028935	0.799999983
0.5	1.0	1.003472222	1.000144676	0.999999916
0.6	1.2	1.197916667	1.200260417	1.199999849
0.7	1.4	1.392361111	1.400376157	1.399999782
0.8	1.6	1.586805556	1.600491898	1.599999715
0.9	1.8	1.781250000	1.800607639	1.799999648
1	2.0	1.975694444	2.000723380	1.999999581
L.S.E.		3.5×10^{-3}	1.704×10^{-6}	5.714×10^{-13}

Table (9) the results for the approximate solution $u_2(x)$ of Example 2, by using SAM

x	Exact value of $u_2(x)$	Approximate value of $u_2(x)$ after		
		3 iterations	6 iterations	12 iterations
0	0	-0.010416667	0.000048225	-0.000000028
0.1	0.03	0.019583333	0.030106096	0.029999939
0.2	0.12	0.109583333	0.120163966	0.119999905
0.3	0.27	0.259583333	0.270221836	0.269999872
0.4	0.48	0.469583333	0.480279707	0.479999838
0.5	0.75	0.739583333	0.750337577	0.749999805
0.6	1.08	1.069583333	1.080395448	1.079999771
0.7	1.47	1.459583333	1.470453318	1.469999738
0.8	1.92	1.909583333	1.920511188	1.919999704
0.9	2.43	2.419583333	2.430569059	2.429999671
1	3.00	2.989583333	3.000626929	2.999999637
L.S.E.		1.2×10^{-3}	1.622×10^{-6}	5.429×10^{-13}

Table (10) the results for the approximate solution u1(x) of Example 2, by using Aitken's on SAM and ADM

x	Exact	Approximate value of $u_1(x)$ by	
		Aitken's on SAM	Aitken's on ADM
0	0	-0.375000000	-0.025000000
0.1	0.2	-0.251041667	0.178648915
0.2	0.4	-1.658333333	0.381784661
0.3	0.6	0.762500000	0.582894737
0.4	0.8	0.827564103	0.441666667
0.5	1.0	1.010416667	1.003787879
0.6	1.2	1.219444444	1.204729730
0.7	1.4	1.438480392	1.407834331
0.8	1.6	1.662398374	1.611472347
0.9	1.8	1.889062500	1.815322581
1	2.0	2.117424242	2.019278607
L.S.E.		4.636	1.309×10^{-1}

Table (11) the results for the approximate solution u2(x) of Example 2, by using Aitken's on SAM and ADM

x	Exact	Approximate value of $u_2(x)$ by	
		Aitken's on SAM	Aitken's on ADM
0	0	-0.083333333	0.013888889
0.1	0.03	0.859166667	0.039369115
0.2	0.12	0.213410853	0.128065596
0.3	0.27	0.331178862	0.277814123
0.4	0.48	0.532203857	0.488036472
0.5	0.75	0.799479167	0.758512545
0.6	1.08	1.129329983	1.089140286
0.7	1.47	1.520490196	1.479865900
0.8	1.92	1.972406739	1.930658346
0.9	2.43	2.484799578	2.441498451
1	3.00	3.057511737	3.012373737
L.S.E.		7.261×10^{-1}	1.1×10^{-3}

Table (12) the result for the approximate solution of u1(x) of Example 2, by using ADM which is derived by (Babolian et al., 2004)

x	Exact value of $u_1(x)$	Approximate value of $u_1(x)$ after		
		3 iterations	6 iterations	12 iterations
0	0	0.005208333	-0.000217014	0.000000126
0.1	0.2	0.204513889	0.199831211	0.200000098
0.2	0.4	0.403819444	0.399879437	0.400000069
0.3	0.6	0.603125000	0.599927662	0.600000041
0.4	0.8	0.802430556	0.799975887	0.800000013
0.5	1.0	1.001736111	1.000024113	0.999999985
0.6	1.2	1.201041667	1.200072338	1.199999957
0.7	1.4	1.400347222	1.400120563	1.399999929
0.8	1.6	1.599652778	1.600168789	1.599999901
0.9	1.8	1.798958333	1.800217014	1.799999873
1	2.0	1.998263889	2.000265239	1.999999845
L.S.E.		8.620×10^{-5}	2.622×10^{-7}	8.916×10^{-14}

Table (13) the result for the approximate solution of $u_1(x)$ of Example 2, by using ADM which is derived by (Babolian et al., 2004)

x	Exact value of $u_2(x)$	Approximate value of $u_2(x)$ after		
		3 iterations	6 iterations	12 iterations
0	0	-0.002314815	0.000048225	-0.000000028
0.1	0.03	0.028032407	0.030062693	0.029999964
0.2	0.12	0.118379630	0.120077160	0.119999955
0.3	0.27	0.268726852	0.270091628	0.269999947
0.4	0.48	0.479074074	0.480106096	0.479999939
0.5	0.75	0.749421296	0.750120563	0.749999930
0.6	1.08	1.079768519	1.080135031	1.079999922
0.7	1.47	1.470115741	1.470149498	1.469999913
0.8	1.92	1.920462963	1.920163966	1.919999905
0.9	2.43	2.430810185	2.430178434	2.429999897
1	3.00	3.001157407	3.000192901	2.999999888
L.S.E.		1.695×10^{-5}	1.829×10^{-7}	6.137×10^{-14}

Table (14) shows the comparison between least square errors for SAM and ADM which is derived by (Babolian et al., 2004) for Example 2

x	3 iterations		6 iterations		12 iterations	
	$u_1(x)$	$u_2(x)$	$u_1(x)$	$u_2(x)$	$u_1(x)$	$u_2(x)$
SAM	3.5×10^{-3}	1.2×10^{-3}	1.704×10^{-6}	1.622×10^{-6}	5.714×10^{-13}	5.429×10^{-13}
ADM	8.620×10^{-5}	1.695×10^{-5}	2.622×10^{-7}	1.829×10^{-7}	8.916×10^{-14}	6.137×10^{-14}

Conclusion

This paper presents the use of successive approximation method, Aitken's method depending on successive approximation method, and Aitken's method depending on Adomian decomposition method for solving linear system of Fredholm integral equation of the second kind. As it can be seen, the successive approximation method is very close (equal, if we choose initial solution $u_{i0}(x)=f_i(x)$ in place of the zero function) to Adomian decomposition method (Babolian et al., 2004), and also we see that the Aitken's method depending on Adomian decomposition method by using the first three function values better than the Aitken's method depending on successive approximation method.

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حل عددي لنظام معادلات تكاملية خطية من النوع الثاني لفريدهولم بطرق التكرارية

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كلية العلوم - جامعة صلاح الدين–أربيل

الخلاصة

فى هذا البحث، تم دراسة نظام معادلات تكاملية خطية من النوع الثاني لفريدهولم. استخدمت ثلاث طرق لحل هذا النظام، طريقة تطابعية تقريبية، طريقة أتكين بالاعتماد على طريقة تطابعية تقريبية و طريقة اخرى والتي هى طريقة أتكين بالاعتماد على طريقة انحلال لادوميان ، تمت مقارنة بين الحلول التقريبية و المضبوطة بمثالين وذلك بالاعتماد على أخطاء المربعات الصغرى(least-square error) وقد أعطيت لتعزيز النتائج التى تم الحصول عليها باستخدام هذه الطرق.